Unfolding Complexity: Hereditory Dynamical Systems - New Bifurcation Schemes and High Dimensional Chaos *

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"The world of difference equations,
which has been almost hidden up to now,
begins to open in all its richness."
A. N. Sharkovsky

Abstract

We discuss some dynamical systems which are paradigmatic for complexity.

The aim of the present paper is twofold: First, to state briefly the state of the art and the problems unanswered until now and, second, to open the way for the discovery of new kinds of complex dynamics. Systems where the future is not only determined by their present state but by part of their history and which formally can be described by seemingly simple difference -differential equations not only play an important role in the applications e. g. of nonlinear delayed feedback but are very suitable for numerical and substantial analytical discussion and insight of complex dynamics. This includes new types of bifurcation patterns, multi–stability of highly structured periodic orbits and high dimensional strange strange attractors.

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1 Introduction: Hereditary systems

The last decades have shown the discovery of complexity in low dimensional dynamical systems. Complexity shows up by phenomena like multiple steady states, limit cycles, extensive bifurcation patterns like period doubling bifurcations, and deterministic chaos ("strange attractors"). Famous examples are the Lorentz attractor, the discrete logistic equation, the Hénon attractor, the equations for the coupled pendula, the Mandelbrot set [3], [30], [33].

Nevertheless and despite of the ongoing difficulties in analyzing them [33], these systems are simple in the sense that in the real world there are only very few things or processes which can be adequately described by two, three or four ordinary differential equations or by one- or two-dimensional maps. E. g. the phenomena of turbulence are generally not yet understood via low dimensional deterministic chaos and form still a major challenge to theoretical physicists and mathematicians.

It is often believed that the now well known bifurcation schemes or scenarios like e.g. pitchfork and saddle node bifurcations, Hopf bifurcation, period doubling bifurcation or bifurcation of chaos from quasi-periodic motion, are generic in the sense that essentially no other bifurcation patterns can generally occur. However, we will indicate that this is far from true opening the way for more possibilities of explaining complex phenomena.

Systems with delays are very suitable for making steps forward to understanding complexity in higher dimensional systems since in a certain sense they lie intermediate between low dimensional ordinary differential equations and systems which must be described by partial differential equations. This may be illustrated by the concrete example of the hyperbolic system

$$\begin{aligned} i_s + Cv_t &= 0 \\ v_s + Li_s &= 0 \end{aligned}$$

with boundary conditions

$$\begin{aligned} v(0,t) &= 0, v(s,0) = v_0(s) \\ i(l,t) &= g(v(l,t)+E), i(s,0) = i_0(s) \end{aligned}$$

modelling electric media with a tunel diode having a non-linear voltage-current characteristic g, for details see [30]. This system can be equivalently transformed into the seemingly simple single equation

$$x(t) = f(x(t-\tau)), t \geq 0, \qquad (1)$$

a so called *difference equation with continuous argument* or *functional difference equation* [30]. The positive constant τ denotes the time needed by a signal to travel from one end of the medium to the other.

Eq.(1) appears to be nearly the same as the *difference equation*

$$x_n = f(x_{n-1}), n \in \mathbb{N} \tag{2}$$

where $f : I \to I$ is a map defined on some interval $I \subset \mathbb{R}$. However, solutions to Eq.(2) and Eq.(1) are drastically different. Eq.(2) may have a strange attractor, the fractal dimension of which, however, is bounded above by 1. The reason is that the state space, i. e. the space of initial conditions, is some subset of $\mathbb{R}$, and thus at most one-dimensional. One example is the famous logistic equation where $f(\xi) = \lambda\xi(1-\xi)$ with a constant parameter $\lambda \in [0, 4]$ and state space the interval [0,1]. For $\lambda = 4$ there is a strange strange attractor with fractal dimension exactly 1; this attractor is dense in the whole state space [0,1].

Contrarily, for Eq.(1) an initial condition is an arbitrary function $\varphi : [-\tau, 0) \to I$, and therefore the state space is $C^{-1}([-\tau, 0), I)$, the space of all functions with domain $[-\tau, 0)$ and range I. Taking again the quadratic function $f(\xi) = \lambda\xi(1-\xi)$ with $\lambda = 4$ there is a strange strange attractor which in fact has dimension ∞.

Before going into details with respect to the already mentioned equations we like to address another, "closely" related type of equations interesting with respect to complicated behavior.

Namely, the equations (1) and (2) have been considered as singular perturbation problems in the context of delay differential equations of type

$$\varepsilon \frac{dx}{dt}(t) + x(t) = f(x(t-1)) \tag{3}$$

in the formal limit $\varepsilon \to 0$.

Eq.(3), the socalled *Mackey-Glass equation*, has found many applications in physics, biology and economics, see e.g. [5]. The reason for the importance of this equation in applications is that it falls under the general scheme where the rate of change dx/dt of some time dependent quantity $x(t)$ is the net effect of two factors, a productive one (p) and a destructive one (d) [15]:

$$\frac{dx}{dt}(t) = p - q.$$

When there is feedback, both p and q may depend on the quantity x itself. Often, the production needs considerable time, e. g. in commodity markets (in particular agricultural ones [22]), in population growth, or in hormonal systems. In such situations the production p may be a functional $p(t) = P(x_t)$ of the history $x_t : (a, 0) \to \mathbb{R}, x_t(s) := x(t+s)$ of the variable x (the constant a being either a negative number or $-\infty$). Similarly with $q = q_t$.

Thus, the most general approach would be to write $x(t)$ as a functional of x_t:

$$x(t) = F(x_t).$$

For most applications the dependence on the past may be made explicit by an integral equation of type

$$x(t) = \int_{-\infty}^{t} f(x(t' - \tau)) g(t - t') dt' \tag{4}$$

where $t \geq 0$, $f : I \to I$, τ a constant delay, and $g : [0, \infty) \to [0, \infty)$ denotes a weighting function (or more generally a distribution).

Taking for g the δ - distribution Eq.(4) becomes Eq.(1) which thus appears as an extreme case of (4).

In applications and for analytical reasons it is useful to consider weighting functions $g = g_k$ of the following type: Let $k \in \mathbb{N}$,

$$g_k(t) := \alpha\, t^{k-1} e^{-\alpha t}/(k-1)!$$

With such a weighting function Eq.(4) can be transformed into a system of differential equations: Define

$$x_i(t) := \int_{-\infty}^{t} f(x(t' - \tau)) g_i(t - t') dt' \quad \text{for} \quad i = 1, 2, \ldots, k, \tag{5}$$

$$x_0(t) := \alpha^{-1} f(x(t - \tau)).$$

Then because of

$$dx_i(t)/dt = \alpha x_{i-1}(t) - \alpha x_i(t) \quad \text{for} \quad i = 1, 2, \ldots, k$$

we arrive at the system

$$dx_1(t)/dt = f(x_k(t - \tau)) - \alpha x_1(t)$$
$$dx_i(t)/dt = \alpha x_{i-1}(t) - \alpha x_i(t) \quad \text{for} \quad i = 2, 3, \ldots \tag{6}$$

Since $g = g_k$ Eq.(4) together with definition (5) implies the identity $x_k = x$, and thus x obeys system (6). Vice versa (4) can be reobtained from (6) by successively integrating the i-th equation,$i = k,\ k-1, \ldots, 1$.

We remark that system (6) represents a feedback model for the regulation of protein synthesis introduced by [6]. In this context system (6) has been studied later on intensively, for a review in case of $\tau = 0$ see [27]. In particular, it could be proved that in case of negative feedback, i. e. if the feedback function f is monotone decreasing, system (6) has non-constant periodic solutions provided that f is bounded and differentiable and the equilibrium is unstable. This result could be generalized to arbitrary $\tau \geq 0$, (case $k = 1$ [7], case $k = 2$ [9], any $k \in \mathbb{N}$ [23], [8]).

By forming the k-th derivative of $x_k(t)$ and using the equations of (5) one can show that system (6), and thus also Eq.(4) with $g = g_k$, is equivalent to the k-th order delay differential equation

$$\sum_{i=0}^{k} \binom{k}{i} \alpha^i x^{(k-i)}(t) = \alpha^k f(x(t-\tau)), \tag{7}$$

where $x^{(i)}$ denotes the i-th derivative of x.

Eq.(7) is a special case of the very interesting class of k-th order delay-differential equations [11]

$$\sum_{i=0}^{k} a_i d^i x(t)/dt^i = f(x(t-\tau)), \quad a_i \in \mathbb{R}. \tag{8}$$

Note that Eq.(1) is obtained from (8) by choosing $k = 0$, and Eq.(3) by choosing k = 1. Formally Eq.(8) is in the "vicinity" of (1), and thus also indirectly connected to Eq.(2), if $a_i = \varepsilon_i$ are small numbers for $i = 1, 2, \ldots, k$, [8], [12]. In particular, the k–th order equation

$$\varepsilon\, d^k x(t)/dt^k + x(t) = f(x(t-\tau)) \tag{9}$$

formally approaches (1) as $\varepsilon \to 0$.

There is a striking contrast between Eqs. (2) and (1) on the one side and Eqs. (3) and (9) on the other side concerning what we know about the complex behavior of solutions and bifurcation patterns. We cite here a statement of A. N. Sharkovsky from 1986: "In spite of the apparent simplicity of Eq.(3), the investigation of it is not an easy task. For any sufficiently small ε, this equation can no longer possess solutions of the turbulent type, because

$$\varepsilon \mid d^k x(t)/dt^k \mid \leq \frac{1}{\varepsilon} \mid -x(t) + f(x(t-\tau)) \mid .$$

Thus, we arrive here at the principal question which still has no answer: What happens with these solutions when $\varepsilon > 0$ and $t \to \infty$? The remark of Sharkovsky "...; unfortunately our understanding of this process leaves much to be desired," ([30], p. 13) still holds even in view of the recent great progress in the analysis of equations like (3), compare [4], [18], [19], [35], [36]. The detailed knowledge about the first class (difference equations) cannot simply be extended by continuity arguments to the second class (differential equations) despite of the formal limit transition $\varepsilon \to 0$ between the two classes. This impossibility has become evident by the work of Mallet-Paret, Nussbaum, and the Russion group around Sharkovsky who showed that there is a "bifurcation gap" between the two classes and that they differ drastically in their asymptotic behavior of solutions ([25],[17]). The problems become already very difficult by the tremendous difference of behavior between (2) and (1), as we already mentioned in the beginning and will describe later on in more detail.

In the following sections the reader is invited onto a pathway of increasing rank and complicatedness of equations like wandering through landscape of different levels with different perspectives and different scenarios. The reader should be stimulated to further considerations by our pointing to areas where there is little insight until now.

2 Difference equations with continuous argument: Idealized turbulence

In this section we briefly recall recent results of Sharkovsky and coworkers concerning equations of type (1), socalled difference equations with continuous argument [30],[32]. The character of their solutions is very much different from that of the difference equation (2) despite of the strict relationship

$$x(t + n\tau) = f^n(x(t)) \text{ for all } n \in \mathbb{N}$$

meaning that for each fixed $t \in [-\tau, \infty)$ the sequence $(x(t + n\tau))_{n\in\mathbb{N}}$ is a solution of (2).

The striking difference can already be seen with the simple example where $f : [-1, 1] \to [-1, 1]$ is given by $f(\xi) = \arctan(\alpha\xi)$ with a constant $\alpha >$

1. With this sigmoid nonlinearity (2) has two attracting fixed points and no periodic solutions, whereas (1) has infinitely many periodic solutions (though it has also two attracting constant solutions). The reader will easily verify that e.g. one of these periodic solutions is obtained by choosing the initial condition $\varphi(t) = t$ for $t \in [-1, 1]$(remember the remark above, concerning initial conditions for (1). The corresponding solution converges, as $t \to \infty$, to a periodic solution which in fact is discontinuous and has a pulstile character. For embedding such solutions into a state space with suitable metric see [30].

In general and following the classification of Sharkovsky et al. [30] Eq.(1) has two main types of solutions:
a)*relaxation type*: smooth, bounded solutions, converging as $t \to \infty$, to discontinuous periodic solutions with finitely many discontinuity points per period,
b)*turbulent type*: smooth, bounded solutions, converging as $t \to \infty$, to "limiting generalized solutions" having infinitely many discontinuities per unit time interval. The frequency of the oscillations on the time interval $[n\tau, (n + 1)\tau]$ increases towards infinity as $n \to \infty$. (One may observe something like this at a smooth shore ("beach") of the ocean when a smooth wave is breaking).

Which type occurs with a given nonlinearity $f : I \to J$ depends on the socalled *separator set* $\mathrm{D}(f) := \{\xi \in I : f^i(\xi), i = 1, 2, \ldots,$ is an unstable trajectory of (2) $\}$. Note that the closure of $\mathrm{D}(f)$ is the *Julia set* of f. A solution of Eq.(1) corresponding to a continuous initial condition $\varphi : [-\tau, 0) \to I$ is of relaxation type if $T := \varphi^{-1}(\mathrm{D}(f))$ is finite, and it is of turbulent type if T is infinite.

[30] distinguishes between three subclasses of the turbulent type:
ba) the *preturbulent type* where T is countable,
bb) the *turbulent type* where T is uncountable, but nowhere dense in I (a "Cantor set"),
bc) the *strong turbulent type* where T contains cyclic intervals with absolutely continuous invariant measures with respect to the map f.
In the preturbulent case the number of oscillations (alternating increase and decrease of $x(t)$) on the time interval $[n\tau, (n + 1)\tau]$ increases according to a power law as $n \to \infty$, however in the turbulent cases bb) and bc) it increases exponentially. Moreover, on the shift sets $nT = \{nt : t \in T\}$ the slopes of solutions tend to infinity (in modulus).

Thus, the classifications a), ba), bb),and bc) are an indication of the enormous richness of solution structures inherent in the deceptively simple Eq.(1), for more details see [31] and [32]. Of course, such an equation is still an extreme caricature of real processes, however the fact that its attractors may be of arbitrary high dimension makes them more suitable for the inherent properties of turbulence than the well known low dimensional strange attractors. It remains a challenge to study these phenomena in detail [32].

There are two intimately related unrealistic features in Eq.(1). First, as $t \to \infty$, solutions become arbitrarily steep on arbitrarily small intervals, such that asymptotically solutions become discontinuous. Second, there is no dissipative or friction term in Eq.(1). Both of these deficits are overcome (at least theoretically) by introducing the friction term $\varepsilon \frac{dx}{dt}(t)$, thus arriving at Eq.(3).

3 First order difference-differential equations: a singular perturbation problem with bifurcation gaps

There are some results which support the belief that for small ε the solutions of Eq.(3) should be very near to the solutions of Eq.(1). E. g. if $\bar{x}$ is an attracting or repelling fixed point of f, then $x(t) = \bar{x}$ is an attracting or repelling constant solution respectively of both (1) and (3). Even the following result holds telling that solutions to Eq.(1) and Eq.(3) can stay arbitrarily close together for arbitrarily, but finitely long times provided the positive ε is small enough.

Proposition 3.1 (Continuous dependence of solutions on the parameter ε for finite time intervals)[17]. *Let $f : I \to I$ be continuous on the closed interval I and let $\varphi : [-\tau, 0] \to \mathbb{R}$ be continuous. Then for each $T > 0$ and for each $\delta > 0$ there is a positive number $\varepsilon^* = \varepsilon^*(T, \varphi, \kappa)$ such that the solution x^0_φ of Eq.(1) and the solution x^ε_φ of Eq.(3) corresponding to the initial condition φ obey*

$$\|x^\varepsilon_\varphi(t) - x^0_\varphi(t)\| < \delta \quad forall \quad t \in T \text{ whenever } \varepsilon < \varepsilon^*.$$

We note that this proposition can be generalized to piecewise continuous f if the number of jumps is finite. However, strange enough, the proposition cannot be generalized to hold also for $T = \infty$. Though solutions to Eq.(3) and Eq.(1) with the same initial condition may stay near to each other for a very long but finite time, asymptotically they can differ substantially, e. g. in one case they may converge to a constant and in the other case to a nonconstant periodic solution with an amplitude independent of ε. Solutions to Eq.(9) can be both simpler and more complicated compared to those of Eq.(1) and also to those of Eq.(2). This may be documented by simple examples as follows.

Example 3.1 *Let the function* $f : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f(\xi) = \begin{cases} a & \text{if} \quad \xi < \Theta \\ b & \text{if} \quad \xi > \Theta \end{cases} \tag{10}$$

where $a, b, \Theta \in \mathbb{R}$ are constant parameters observing $a \neq b$.

Note that non - smooth nonlinearities like those in this example occur in dynamical systems where one of the variables can only attain a finite number of discrete values. Examples are heating or cooling machines which are either "on" or "off", electric circuits with relays which are "open" or "closed", neurons which are either "firing" or "silent". In fact, in some types of *artificial neural networks* Eq.(3) with f as in Eq.(10) are used for model neurons [28]. Another reason for studying systems with non-smooth nonlinearities is that they often allow for much more and much easier mathematical analysis and provable results than in case of smooth nonlinearities. Often solutions can be explicitly and exactly calculated by being piecewise composed of solutions of linear systems.

Proposition 3.2 (Infinitely many unstable high frequency periodic solutions) *Consider Eq.*(3) *with* $\varepsilon > 0$ *and* f *given by* (10) *with* $a \neq b$. *Without loss of generality assume* $\Theta = 0$. *Then Eq.*(3) *has infinitely many periodic solutions with pairwise different minimal periods. The countable set of periods can be written as a sequence converging to zero. In case of "positive feedback"* ($b > 0 > a$) *all of these periodic solutions are unstable. In case of "negative feedback"* ($a > 0 > b$) *the periodic orbit with the largest minimal period is asymptotically orbitally stable, all other periodic orbits are unstable.*

A proof of this proposition was given by [1], see also [4], [17]. [1] could not only show that most of the periodic solutions are unstable, but that the asymptotic behavior of almost all solutions is very simple:

Proposition 3.3 *Let the assumptions of Proposition 3.2 hold.*

(i) Let $a < 0 < b$. *Then almost all solutions of* (3) *satisfy* $\lim_{t\to\infty} x(t) = a$ *or* $\lim_{t\to\infty} x(t) = b$.

(ii) Let $b > 0 > a$. Then Eq.3 has an asymptotically stable periodic solution (with period $> 2\tau$) and almost all solutions converge towards this periodic solution (of course in the sense of orbital convergence).

Here "almost all" means that the corresponding set of initial conditions is open and dense in the state space $C([-1,0],\mathbb{R})$. A result similar to Prop.3.3 has been obtained by [35] for the equation

$$dx(t)/dt = f(x(t-1)) \tag{11}$$

for a rather general class of *continuous* functions f satisfying the condition $xf(x) < 0$ if $x \neq 0$.

Contrarily to Proposition 3.3, which says that Eq.(3) possesses at most two stable orbits, Eq.(1) with the nonlinearity f satisfying (10) has infinitely many asymptotically stable periodic solutions with pairwise different minimal periods; note however, that the corresponding stable orbits lie in the extended state space $\mathrm{PC}([-\tau,0),I)$ of piecewise constant functions $\varphi : [-\tau,0) \to I$.

Mixed feedback and chaos

In case of monotone feedback functions f the phenomenon of deterministic chaos seems to be excluded. This is apparent for difference equations like (2) and has been shown for first order difference-differential equations by [24]. There is a long history of investigations concerning the chaotic behavior of solutions to difference equations (2) if f is non-monotone, e.g. in case of the discrete logistic equation where $f(x) = \lambda x(1-x)$ with a constant λ. If one tries to prove existence of chaos for ordinary differential equation systems or difference-differential equations like (3) results are not easily obtained. One of the earliest successes were presented by [34], [16] and [15] under the simplifying assumption that f is piecewise constant or, though smooth, near to a piecewise constant function. Recently [18], [19] succeeded to prove existence of chaos also for smooth nonlinearities, at least in equations of type (11). Here we just summarize some results with respect to non-smooth functions f defined as following:

$$f(\xi) = \begin{cases} 0 & \text{if} \quad \xi < 1 \\ c & \text{if} \quad 1 < \xi < \Theta \\ d & \text{if} \quad \xi > \Theta \end{cases} \tag{12}$$

where the constants are assumed to obey

$$c > 0, \quad \Theta > 1, \quad d < c. \tag{13}$$

We introduce the following notion of chaos which is adapted from the definition which [20] gave for difference equations.

Definition 3.1 *A difference-differential equation of type* (2) *is called* ***chaotic in the sense of Li and Yorke*** *("Li-Yorke-chaotic") if*

(i) *there are countably many periodic solutions with pairwise different minimal periods*

(ii) *there is an uncountable set S of aperiodic solutions such that*

(iia) *if x is a periodic solution and $\tilde{x} \in S$ then* $\limsup_{t\to\infty} \|x_t - \tilde{x}_t\| > 0$

(iib) *if $x, \tilde{x} \in S$ and $x \neq \tilde{x}$ then*
$\liminf_{t\to\infty} \|x_t - \tilde{x}_t\| = 0$ *and* $\limsup_{t\to\infty} \|x_t - \tilde{x}_t\| > 0$.

Here, as usual, $x_t : [-\tau, 0] \to \mathbb{R}$ is the shift function defined by $x_t(s) = x(t+s)$, and $\|x_t\| := \sup\{|x_t(s)| : s \in [-\tau, 0]\}$.

The following theorem has been proved by [15] after the precursor [16] where a more complicated f (with 3 discontinuities) has been used.

Theorem 3.1 *Let the function f be defined by* (12) *with the parameters c, Θ and d obeying* (13). *Assume, moreover, that ε and c satisfy*

$$c/(c-1)^2 + z < 1$$

where z is the positive root of the quadratic

$$z^2 - (c - r - c^2)z - cr = 0$$

with $r := (c-1)\exp(-1/\varepsilon)$.
Then there are positive numbers $\mu = \mu(c, \varepsilon)$ and $d^* = d^*(c, \varepsilon)$ such that Eq.(3) *is chaotic in the sense of Li and Yorke whenever Θ and d satisfy*

$$(c-z)/(c-1) < \Theta < (c-z)/(c+1) + \mu \quad \text{and} \quad d \leq d^*.$$

It is remarkable that, with f given by (12), neither Eq.(2) nor Eq.(1), which is the limiting case of Eq.(3) for $\varepsilon = 0$, exhibit this kind of chaos, which may happen for arbitrarily small, but positive ε.

For further results about chaos in first order differential delay equations see [18], [19], [17].

4 Prime number dynamics of a retarded difference equation

If one tries to solve the difference equation with continuous argument (1) by help of a digital computer the simplest way would be to discretize time t by discrete values $t_n = nh, n \in \mathbb{N}$ with some fixed positive step size h. If one takes $h = \tau/k$ then the resulting discretization of Eq.(1) is [12]

$$x_n := x(t_n) = f(x(t_n - \tau)) = f(x(nh - kh) = f(x((n-k)h)) = f(x_{n-k}),$$

i.e.

$$x_n = f(x_{n-k}), \quad n \in \mathbb{N}. \tag{14}$$

Note that the discrete approximation (14) is independent of the value of τ, corresponding to the fact that without loss of generality in Eq.(1) one can assume $\tau = 1$.

When $k > 1$ Eq.(14) is called a *retarded difference equation*. Of course, with $k = 1$ we arrive again at Eq.(2) which, viewed this way, is a crude discretization of Eq.(1).

While lying somehow in between Eq.(2) and Eq.(1) it will be of interest of what kind the relation of Eq.(14) is to Eqs. of type (1), (2) or (3). Before we come back to this question we will show first that the structure of solutions of Eq.(14) may be considerably more complex in case of $k > 1$ than in case $k = 1$.

One of the deepest results concerning the case $k = 1$ is the following one by A. N. Sharkovsky which includes the famous period doubling bifurcation and "period three implies chaos":

Theorem 4.1 [Sharkovsky 1964] *Let $f : I \to \mathbb{R}$ be a continuous function defined on some interval $I \subseteq \mathbb{R}$. Let the set $\mathbb{N}$ of natural numbers be ordered in the following way*

$$3 \triangleright 5 \triangleright \cdots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright \cdots \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright \cdots \triangleright 2^2 \triangleright 2 \triangleright 1. \tag{15}$$

If the equation $x_n = f(x_{n-1})$ has a cycle with period p then it has also cycles with period p' for all p' positioned to the right of p in the ordering (15).

In the "Sharkovsky ordering" (15) each natural number appears exactly once, and since 3 is the most left number this theorem tells in particular that if there is a periodic solution with period 3 then also for each $p \in \mathbb{N}$ a periodic solution with period p does exist (compare [20]).

We have generalized Theorem 4.1 to cover Eq.(14) for all $k \in \mathbb{N}$ [14].

For arbitrary $k \in \mathbb{N}$ the state space of Eq.(14) is the k-dimensional cube I^k (remember $f : I \to I$)and an initial condition is a vector $x^{(0)} = (x_{-k}, x_{-k+1}, \ldots, x_{-1}) \in I^k$. A *solution* of Eq.(14) corresponding to the initial condition $x^{(0)}$ is a sequence (x_n) with $n \in \mathbb{N} \cup \{-k+1, \ldots, 0\}$, satisfying Eq.(14) for all $n \in \mathbb{N}$ and obeying $(x_{-k+1}, \ldots, x_0) = x^0$. A solution (x_n) of Eq.(14) is called *periodic* with period $p \in \mathbb{N}$ if

$$x_{n+p} = x_n \quad \text{for all } n \in \mathbb{N}.$$

Theorem 4.2 *(an der Heiden & Liang [14]) Let $f : I \to J$ be a continuous function defined on some interval $I \subset \mathbb{R}$ and with range $I \subset \mathbb{R}$. Let $k \in \mathbb{N}$. If the difference equation*

$$x_n = f(x_{n-1})$$

has a cycle of minimal period p then the difference equation

$$x_n = f(x_{n-k}) \tag{16}$$

has cycles with minimal period p' for all numbers $p' \in S_k(m)$ whenever $S_k(m)$ is either equal to $S_k(p)$ or to the right of $S_k(p)$ in the "Sharkovsky sequence of order k" defined by

$$S_k(3) \triangleright S_k(5) \triangleright \cdots \triangleright S_k(2 \cdot 3) \triangleright S_k(2 \cdot 5) \triangleright \cdots \triangleright S_k(2^2 \cdot 3) \triangleright S_k(2^2 \cdot 5) \triangleright$$
$$\triangleright \cdots \triangleright S_k(2^2) \triangleright S_k(2) \triangleright S_k(1).$$

Here $S_k(p)$ denotes a certain set given by

$$S_k(p) := \begin{cases} \{1\} & \text{if } \ p = 1 \\ \{l \cdot p \mid l \in \mathbb{N}, l \text{ divides } k \text{ and } (\frac{k}{l}, p) \text{ coprime } \} & \text{for } \ p \in \mathbb{N} \backslash \{1\} \end{cases}$$

If f has more than one fixed point then Eq.(16) has also cycles with minimal period p' for all $p' \in S_k(*) := \{l \mid 2 \leq l \leq k, l \text{ divides } k\}$.

Of course, a pair(m, n) of natural numbers is called coprime if 1 is the only common divisor of m and n. For illustration of this theorem we choose the following example.

Example 4.1 *Let $k \in \mathbb{N}$ be a prime number. Then l divides k if and only if $l = 1$ or $l = k$; moreover $(\frac{k}{l}, p)$ is coprime if and only if p is not a multiple of k. Thus for $p \in \mathbb{N}$, $p > 1$ we have*

$$S_k(p) = \begin{cases} \{kp\} & \text{if } p \text{ is a multiple of } k \\ \{p, kp\} & \text{if } p \text{ is not a multiple of } k. \end{cases}$$

Thus, e.g. the Sharkovsky sequence of order $k = 11$ is

$$\{3, 33\} \triangleright \{5, 55\} \triangleright \ldots \triangleright \{9, 99\} \triangleright \{111\} \triangleright \{13, 143\} \triangleright \ldots \triangleright \{2 \cdot 3, 2 \cdot 3 \cdot 11\}$$
$$\triangleright \ldots \triangleright \{4, 44\} \triangleright \{2, 22\} \triangleright \{1, 11\}.$$

It follows from this ordering that if $x_n = f(x_{n-1})$ has for example a period 4 cycle, then $x_n = f(x_{n-11})$ has cycles with minimal period $1, 11, 2, 22, 4,$ and 44.

The following theorem tells something about the number of periodic orbits ("cycles") of (16).

Theorem 4.3 *(an der Heiden & Liang [14]) Let $M, N \subset \mathbb{R}$, $f : M \to N$, and let $S = \{s_1, \ldots, s_p\}$ be a cycle of minimal period p of the difference equation $x_n = f(x_{n-1})$.*

Let $k \in \mathbb{N}\backslash\{1\}$. Then the number $\mathcal{N}(p, k)$ of periodic cycles of the difference equation

$$x_n = f(x_{n-k})$$

which obey $x_n \in S$ for all $n \in \mathbb{N}$ is exactly given by

$$\mathcal{N}(p, k) = \frac{1}{p} \cdot \sum_{i \in A_k^{(p)}} \frac{p^i}{i} \Upsilon(\frac{k}{i}),$$

where $A_k^{(p)} := \{i \in \mathbb{N} \mid \ i \text{ divides } k \text{ and } (k/i, p) \text{ coprime } \}$,

$$\Upsilon(m) := \begin{cases} 1 & \text{if } m = 1, \\ \prod_{\iota=1}^{\kappa} \frac{m_\iota - 1}{m_\iota} & \text{if } m \in \mathbb{N}\backslash\{1\} \text{ and } \{m_1, \ldots, m_\kappa\} \\ & \quad \text{is the set of pairwise different prime factors of } m. \end{cases}$$

Example 4.2 *Let $k = 11$ and $x_n = f(x_{n-1})$ have a 4–cycle $\{s_1, s_2, s_3, s_4\}$. Then $A_k^{(p)} = A_{11}^{(4)} = \{i \in \mathbb{N} : i \text{ divides } 11 \text{ and } (\frac{11}{i}, 4) \text{ coprime } \} = \{1, 11\}$.*

Thus

$$\mathcal{N}(p,k) = \mathcal{N}(4,11) = \frac{1}{4}(\Upsilon(11) + \frac{4^{11}}{11}\Upsilon(1)) = \frac{10}{11} + \frac{4^{10}}{11}$$
$$= 95326 = (\textit{ number of periodic cycles of } x_n = f(x_{n-11})$$
$$\textit{with } x_n \in \{s_1, s_2, s_3, s_4\} \textit{ for all } n \in \mathbb{N}).$$

Proposition 4.1 *Let the equation $x_n = f(x_{n-1})$ have a strange attractor with Hausdorff dimension H. Then for each $k \in \mathbb{N}$ the equation $x_n = f(x_{n-k})$ has a strange attractor with Hausdorff dimension $k \cdot H$.*

5 Second order non–smooth difference–differential equations

We now turn to the case $k = 2$ of Eq.(9):

$$d^2x(t)/dt^2 = f(x(t-\tau)) - \alpha x(t), \tag{17}$$

where $\alpha > 0$.

Without loss of generality we assume $\alpha = 1$. Then (17) can be transformed into the system

$$\begin{array}{rcl} dx(t)/dt &=& y(t) \\ dy(t)/dt &=& f(x(t-\tau)) - x(t). \end{array} \tag{18}$$

Let us first consider the situation of non-smooth feedback with f defined by (10). Without loss of generality $a = 1/2$ and $b = -1/2$, thus there remain just the two parameters τ and Θ. An initial condition of (18) is a pair $(\varphi, y_0) \in C^1([-\tau, 0], \mathbb{R}) \times \mathbb{R}$ such that $\varphi'(0) = y_0$. Solutions $(x(t)), y(t)), t \geq 0$, of (18) can be represented as continuous trajectories $t \to (x(t), y(t))$ in the x-y-plane ($\mathbb{R}^2$). If f is of type (10) and if, moreover, the set $\{t : \varphi(t) = \Theta\}$ is finite then these trajectories are piecewise composed of arcs ("sectors") of circles having center at either $(a, 0)$ or $(b, 0)$, e.g. the center is $(a, 0)$ for all $t \in [t_1, t_2]$ if for all $t \in (t_1 - \tau, t_2 - \tau)$ the inequality $x(t) < \Theta$ is satisfied. Note that the angular length of the arc associated with $[t_1, t_2]$ is just $t_2 - t_1$, because the angular velocity of the trajectory point $(x(t), y(t))$ is always 1, independent of the varying radii of the arcs which make up the trajectory. For the situation of negative feedback the following theorem, a proof of which can be found in [2], shows that for $k = 2$ in (9) there are substantially more periodic solutions than in the case $k = 1$ in (9).

Theorem 5.1 *Let f be given by* (10) *and let $a = 1/2$ and $b = -1/2$ (which can be assumed without loss of generality whenever $b < a$). Let $\Theta \in [0, 1/2]$. Then*

(i) *for each $n \in \mathbb{N}$ and for each $\tau \in (0, 2n\pi)$ System* (18) *(and thus also Eq.*(17)*) has a periodic solution with minimal period τ/n.*

(ii) *for each $n \in \mathbb{N}$, n odd, and for each $\tau \in (n\pi, 2n\pi)$ System* (18) *(and thus also Eq.*(17)*) has periodic solutions with minimal period $2\tau/n$.*

Multistability of periodic orbits In [2] a bifurcation scheme for these periodic solutions is described, and it is shown that, contrary to the first order case ($k = 1$) in Eq.(9), there may coexist more than one asymptotically orbitally stable periodic orbit for fixed values of the parameters τ and Θ.

Chaos. We still consider System (18), however now with a mixed feedback nonlinearity given by

$$f(\xi) = \begin{cases} a & if \quad \xi < \Theta_1 \\ b & if \quad \Theta_1 < \xi < \Theta_2 \\ c & if \quad \xi > \Theta_2 \end{cases} \tag{19}$$

with constants $\Theta_1 < \Theta_2$, $a < b$, $c < b$. Without loss of generality we assume that

$$\alpha = 1,\ \Theta_1 = 0,\ a = -1/2,\ b = 1/2,\ c < 1/2. \tag{20}$$

Hence, the only free parameters are τ, Θ_2, and c.

A proof of the following theorem can be found in [13].

Theorem 5.2 *Let the function f be given by (19). Without loss of generality let the parameters α, Θ_1, a, b, and c satisfy* (20). *Let, moreover, the parameters τ and Θ_2 obey the conditions*

$$0 < \tau < \pi/2,$$

$$(1 + \cos\frac{\tau}{2})/2 < \Theta_2 < 1,$$

$$1 + (2\Theta_2 - 1)\cos\tau > 2\sqrt{\Theta_2(1 - \Theta_2)}\sqrt{\sin^2\tau + \frac{1 - \cos\tau}{1 + \cos\tau}}.$$

Then there are numbers $c_1 = c_1(\tau, \Theta_2)$ and $c_2 = c_2(\tau, \Theta_2)$, $c_1 < c_2$, such that Equation (17) is chaotic in the sense of Li and Yorke whenever the parameter c satisfies $c_1 < c \leq c_2$.

Thus, we have learned that the equation (3) displays different realms of behavior for $k = 0, 1$, and 2, no matter how large ε is. For $k > 2$ nearly nothing is known.

Outlook: Though we have definitely not covered all of the literature about the kind of equations considered here, it is hoped that nevertheless the reader has got the impression that the pathway we followed opens the perspective to many further very interesting investigations with the promise that, in the spirit of Sharkovsky's word at the beginning, there can be discovered a still much richer realm of fascinating phenomena than we already know. In particular many new forms of bifurcations patterns and of high dimensional strange attractors lie ahead.

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References

[1] Aliev, S. Y., Ivanov, A. F., Maistrenko, Y. L., Sharkovski, A. N., Singular perturbations of difference equations with continuous time. Preprint no. 84.33. Inst. of Math. of the Ukrainian Acad. Sciences, 1984 (Russian)

[2] Bayer, W., an der Heiden, U., Oscillation types and bifurcations of nonlinear second order differential-difference equations. J. Dynamics & Diff. Eqs. 10, 303–326 (1998)

[3] Collet P., Eckmann J.–P., Iterated Maps on the Intervals as Dynamical Systems, Birkhäuser, Boston, 1980.

[4] Fridman, L. M., Fridman, E. M., Shustin, E. I., Steady modes in an autonomous system with break and delay. Differential Equations 29, 1161–1166 (1993)

[5] Glass, L., Mackey, M.C., From Clocks to Chaos, Princeton University Press, Princeton 1988

[6] Goodwin, B. C., In "Advances in Enzyme Regulation",(G. Weber ed.), Vol. 3, p. 425, Pergamon Press, Oxford 1965

[7] Hadeler K. P., Tomiuk J., Periodic solutions of difference–differential equations, Arch. Rational Mech. Anal. 65, 87–95 (1977)

[8] Hale, J. K., Ivanov, A. F., On higher order differential delay equations, Math. Anal. App. 173, 505-514 (1993)

[9] an der Heiden, U., Periodic solutions of a nonlinear second order differential equation with delay, J. Math. Anal. & Appl. 70, 599–609 (1979)

[10] an der Heiden, U., Periodic, aperiodic, and stochastic behavior of differential-difference equations modeling biological and economical processes, pp. 91-108 in "Differential-Difference Equations" (L. Collatz, G. Meinardus, W. Wetterling eds.), Birkhaeuser, Basel-Boston 1983, pp. 91-108

[11] an der Heiden, U., Longtin, A., Mackey, M.C., J. Milton, J. Scholl, R.: Oscillatory modes in a nonlinear second order differential equation with delay. J. of Dynamics and Differential Equations 2, 423-449 (1990)

[12] an der Heiden, U., Nonlinear feedback systems with memory: From 0-th to higher order, discrete and continuous, Fields Institute Communications 20, 223–234 (2001)

[13] an der Heiden, U., Bayer, W., Chaos proved for a second order difference–differential equation, Nonlinear Analysis – Theory, Methods & Applications 48, 464–474 (2001)

[14] an der Heiden, U., Liang, M.–L., Sharkovsky – sequences of higher order difference equations, (submitted)

[15] an der Heiden, U., Mackey, M.C., The dynamics of production and destruction: Analytic insight into complex behavior, J. Math. Biol. 16), 75–101 (1982)

[16] an der Heiden, U., Walther, H. O. Existence of chaos in control systems with delayed feedback. J. Diff. Eqs. 47, 273–295 (1983)

[17] Ivanov, A. F., Sharkovsky, A. N., Oscillations in singularly perturbed delay equations, pp. 164-224 in "Dynamics Reported - Expositions in Dynamical Systems", Vol. 1 (C. K. R. T. Jones, U.Kirchgraber, H. O. Walther eds.), Springer-Verlag, Berlin Heidelberg New York 1992

[18] Lani–Wayda, B., Walther, H. O., Chaotic motion generated by delayed negative feedback. Part I: A transversality criterion, Diff. Int. Eqs. 8, 1407–1452 (1995)

[19] Lani–Wayda, B., Walther, H. O., Chaotic motion generated by delayed negative feedback. Part II. Construction of nonlinearities, Mathemat. Nachrichten 180, 141–211 (1996)

[20] Li, T.Y., Yorke, J.A., Period three implies chaos, Americ. Math. Monthly 82, 985–992 (1975)

[21] Liang, M.–L., an der Heiden, U., Multistability of the difference equation $x_n = f(x_{n-k})$ with an application to prime number factorization of large natural numbers (working title, in preparation)

[22] Mackey, M. C., Commodity fluctuations: Price dependent delays and nonlinearities as explanatory factors, J. Econ. Theory 48, 497–509 (1989)

[23] Mahaffy, J. M., Periodic solutions for certain protein synthesis models, J. Math. Anal. Appl. 74, 72–105 (1980)

[24] Mallet–Paret, J., Smith, H., The Poincaré-Bendixson theorem for monotone cyclic feedback systems, J. Dyn. and Diff. Eqs. 2, 367–421 (1990)

[25] Mallet–Paret, J., Nussbaum, R. D., A bifurcation gap for a singularly perturbed delay equation, pp. 263-286 in "Chaotic Dynamics and Fractals" (F. D. Barnsley, S. G. Demko eds.), Academic Press,New York 1986

[26] Nussbaum, R. D., Mallet-Paret, J., Global continuation and asmptotic behavior of periodic solutions of a differential delay equation. Ann. Mat.Pura Appl. 145, 33–128 (1986)

[27] Othmer, H. G., The qualitative dynamics of a class of biochemical control circuits, J. Math. Biol. 3, 53–78 (1976)

[28] Ritter H., Martinetz, T., Schulten K., Neuronale Netze, Bonn/New York, Addison–Wesley, 1991

[29] Sharkovsky, A. N., Coexistence of cycles of a continuous transformation of a line into itself, Ukrain. Mat. Zhurn. 16, 61–71, 1964 (Russian)

[30] Sharkovsky, A. N., Maistrenko, Y. L., Romanenko, E. Y., Difference Equations and their Applications, Nauka Dumka, Kiev 1986 (Russian), Kluwer 1993 (English)

[31] Sharkovsky, A. N., Kolyada, S. F., Sivek, A. G., Fedorenko, V. V., Dynamics of One - Dimensional Maps, Nauka Dumka, Kiev 1989 (Russian), Kluwer 1997 (English)

[32] Sharkovsky, A. N., Romanenko E. Y., Vereikina, M. B., Ideal Turbulence – Structures, Self-Similarity and Random Evolution Birth in Deterministic Systems, World Scientific, London 1999

[33] Viana, M., What's new on Lorenz strange attractors? The Math. Intelligencer 22, 6–19 (2000)

[34] Walther, H. O., Homoclinic solution and chaos of $\dot{x}(t) = f(x(t-1))$, Nonlinear Anal., Theory, Methods and Appl. 5, 775–788 (1981)

[35] Walther, H. O., Density of slowly oscillating solutions of $\dot{x}(t) = -f(x(t-1))$, J. Math. Anal. Appl. 79, 127–140 (1981)

[36] Walther H. O., The 2–dimensional atractor of $x'(t) = -\mu x(t) + f(x(t-1))$, Memoirs of the Amer. Math. Soc. 113, # 544 (1995)